CROSSCAP NUMBERS OF PRETZEL KNOTS

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Abstract. The crosscap number of a knot in the 3-sphere is defined as the minimal first Betti number of non-orientable subsurfaces bounded by the knot. In this paper, we determine the crosscap numbers of a large class of the pretzel knots. The key ingredient to obtain the result is the algorithm of enumerating all essential surfaces for Montesinos knots developed by Hatcher and Oertel.

1. Introduction

In the 3-sphere S^3, every knot bounds an orientable surface, called a *Seifert surface*. The *genus* $g(K)$ of a knot K is then defined as the minimal genus of Seifert surfaces for K. On the other hand, any knot in S^3 also bounds a non-orientable subsurface in S^3: For, at least one of the two checkerboard surfaces for a diagram of the knot must be non-orientable. In view of this, similarly to the genus of a knot, B.E. Clark defined the crosscap number of a knot as follows.

Definition 1.1 (Clark,[C78]). The *crosscap number* $\mathrm{Crosscap}(K)$ of a knot K is defined as the minimal first Betti number of non-orientable subsurfaces bounded by K in S^3.

For completeness, we define $\mathrm{Crosscap}(K) = 0$ if and only if K is the unknot.

In general, it is rather difficult to decide the crosscap number for a given knot. As far as the authors know, results on the determination of crosscap numbers are only as follows: A knot has crosscap number one if and only if it is $(2, n)$-cabled, shown by Clark [C78]. The knot 7_4 in the knot table has crosscap number three; $\mathrm{Crosscap}(7_4) = 3$, proved by Murakami and Yasuhara [MY95]. This gives the first example showing that the inequality; $\mathrm{Crosscap}(K) \leq 2g(K)+1$, presented by Clark, is sharp. In [T01], Teragaito determined the crosscap number of genus one knots. Next, in [T04], he gave a formula for the crosscap numbers of torus knots. Recently, for two-bridge knots, Hirasawa and Teragaito established a practical algorithm to calculate their crosscap numbers in [HT].

In this paper, we determine the crosscap numbers of a large class of the pretzel knots. Let $K = P(p_1, p_2, \dots, p_N)$ be a non-trivial pretzel knot constructed from N rational tangles corresponding to $(1/p_1, 1/p_2, \dots, 1/p_N)$, where each p_i denotes an integer and

(*) the integer p_i is other than $0, \pm 1$.

It is easily seen that, on the tuple $(p_1, p_2, \dots, p_N)$, either of the following conditions must be satisfied for K to become a knot (not a link);

(a) among $p_1, p_2, \dots, p_N$, exactly one of them is even and the others are odd,

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(b) N is odd and all of $p_1, p_2, \ldots, p_N$ are odd.

Then, our result is stated as follows.

Theorem 1.2. *The crosscap number of a non-trivial pretzel knot* $K = P(p_1, p_2, \ldots, p_N)$ *satisfying (*) is determined as*

$$\mathrm{Crosscap}(K) = \begin{cases} N-1 & \text{when } (a) \text{ is satisfied,} \\ N & \text{when } (b) \text{ is satisfied.} \end{cases}$$

The key ingredient to prove the theorem is the algorithm of enumerating all essential surfaces for Montesinos knots, which are knots obtained by connecting a number of rational tangles in line, developed by Hatcher and Oertel in [HO89]. In the next section, we will collect basic notions, review their algorithm briefly, and give some supplements about the main theorem. The main theorem will be proved in Section 3.

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2. Preliminary

In this section, we will prepare terminologies used in the rest of the paper, briefly review the algorithm of Hatcher and Oertel, and give supplements to our main theorem.

2.1. **Definitions.** For a knot K, there is a connected, possibly non-orientable subsurface embedded in S^3 which has K as the boundary. We call the intersection of the surface and the exterior of the knot a *spanning surface* for K. It is a compact connected surface embedded in the exterior of the knot.

An embedded surface F in a knot exterior is called *incompressible* if F has no compression disks, *boundary incompressible* if F has no boundary-compression disks, and *essential* if F is incompressible and boundary-incompressible. The boundary ∂F of F may intersect the meridian of K more than once. The minimal number of such intersection points is called the *number of sheets* of F, and is denoted by $\sharp s(F)$ or simply by $\sharp s$. In the following, the first Betti number and the Euler characteristic of the surface F are denoted by $\beta_1(F)$ and $\chi(F)$.

2.2. **The algorithm of Hatcher and Oertel.** Here we give a simple review of the algorithm of Hatcher and Oertel, mainly for pretzel knots. Please refer [HO89] or [IM] for details.

The algorithm assumes that the knot K is a Montesinos knot, has no integer tangles, and has 3 or more rational tangles. For the Montesinos knot K, the algorithm enumerates all boundary slopes of essential surfaces in the exterior of K. In their algorithm, embedded surfaces are expressed by "edgepath systems" on a particular 1-dimensional cellular complex, called "diagram" $\mathcal{D}$, lying on the u-v plane. An edgepath system Γ is composed of "edgepaths" $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$, and each "edgepath" γ_i represents subsurfaces around the i-th tangle of K.

The diagram $\mathcal{D}$ is a graph on the uv-plane lying in the region $-1 \le u \le 1$. Vertices of $\mathcal{D}$ are classified into three types: a vertex $\langle p/q \rangle$, a vertex $\langle p/q \rangle^\circ$ and a vertex $\langle 1/0 \rangle$, where p/q is an arbitrary irreducible fraction. Their coordinates are $(u, v) = ((|q|-1)/|q|, p/q)$, $(1, p/q)$ and $(-1, 0)$ respectively. For p/q and r/s,

if $|ps - qr| = 1$, vertices $\langle p/q\rangle$ and $\langle r/s\rangle$ are connected by an edge denoted by $\langle p/q\rangle - \langle r/s\rangle$. In particular, an edge of the form $\langle z\rangle - \langle z+1\rangle$ for some integer z is called a *vertical edge*. Besides, there are edges of the form $\langle p/q\rangle - \langle p/q\rangle^\circ$, which are called *horizontal edges*. An edge of the diagram $\mathcal{D}$ is also called a *complete edge*. In contrast, a segment on a complete edge which is a proper subset of the complete edge as a set is called a *partial edge*.

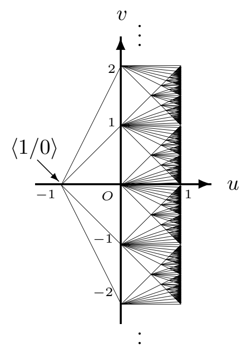

FIGURE 1. The diagram $\mathcal{D}$

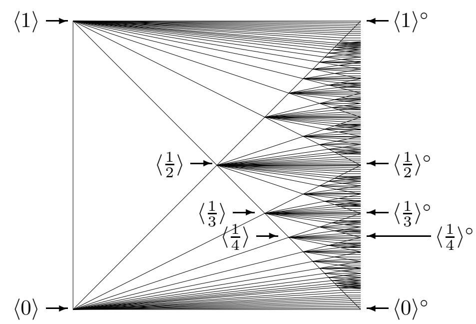

FIGURE 2. A part of the diagram $\mathcal{D}$ in $[0,1]\times[0,1]$

In the algorithm, for a fixed Montesinos knot, we first enumerate all "basic edgepath systems". A *basic edgepath system* is a collection $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of N basic edgepaths. As for a pretzel knot satisfying (*), each *basic edgepath* λ_i is either of $\lambda_{i,a} = \langle 0\rangle - \langle 1/p_i\rangle$ or $\lambda_{i,b} = \langle s_i\rangle - \langle s_i/2\rangle - \ldots - \langle s_i/(|p_i|-1)\rangle - \langle s_i/|p_i|\rangle$, where s_i denotes the sign $+1$ or -1 of p_i. Basic edgepaths $\lambda_{i,a}$ and $\lambda_{i,b}$ are extended to $\widetilde{\lambda_{i,a}} = \langle 0\rangle - \langle 1/p_i\rangle - \langle 1/p_i\rangle^\circ$, and $\widetilde{\lambda_{i,b}} = \langle s_i\rangle - \langle s_i/2\rangle - \ldots - \langle s_i/(|p_i|-1)\rangle - \langle s_i/|p_i|\rangle - \langle s_i/|p_i|\rangle^\circ$ which are called *extended basic edgepaths*. In the same way, we can extend a basic edgepath system $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ to an extended basic edgepath system $\widetilde{\Lambda} = \{\widetilde{\lambda_1}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_N}\}$. An extended basic edgepath $\widetilde{\lambda}$ can be regarded as a function $[0,1] \to \mathbb{R}$ which returns v_0 for u_0 when $\widetilde{\lambda}$ meets with the vertical line $u = u_0$ at (u_0, v_0), and moreover, an extended basic edgepath system can be regarded as a function $[0,1] \to \mathbb{R}$ defined by $\widetilde{\Lambda}(u) = \sum_{i=1}^{N} \widetilde{\lambda}_i(u)$.

Then, next in the algorithm, we enumerate *candidate edgepath systems* of type I, II or III. (I) For each basic edgepath system Λ, we solve the equation $\widetilde{\Lambda}(u) = 0$. Then, for each solution u_0, we make an edgepath system $\Gamma = \{\gamma_i\}$ as follows. If $u_0 \le (|p_i|-1)/|p_i|$, then we set $\gamma_i = \lambda_i \cap \{(u,v) | u \ge u_0\}$, where a partial edge may be included in γ_i. If otherwise, then we set $\gamma_i = \{P_i\}$ where P_i is a point with uv-coordinate $(u,v) = (u_0, 1/p_i)$ lying on a horizontal edge $\langle 1/p_i\rangle - \langle 1/p_i\rangle^\circ$. This edgepath is called a *constant edgepath*. An edgepath system thus defined is called a *type I edgepath system*. (II) For a basic edgepath λ, by adding vertical edges to λ, we can make an edgepath, for instance, $\langle -2\rangle - \langle -1\rangle - \langle 0\rangle - \langle 1/p_i\rangle$ or $\langle s_i + 1\rangle - \langle s_i\rangle - \langle s_i/2\rangle - \ldots - \langle s_i/(|p_i|-1)\rangle - \langle s_i/|p_i|\rangle$. Such an edgepath and a basic edgepath itself are called *type II edgepaths*. For each basic edgepath system Λ, we make an edgepath system Γ that each edgepath γ_i of Γ is a type II edgepath for $1/p_i$ and that v-coordinates of endpoints of the edgepaths in Γ sum up to 0. Such an edgepath system is called a *type II edgepath system*. (III) For a basic edgepath

λ_i, we make an edgepath $\gamma_i = \langle 1/0 \rangle - \lambda_i$, which is called a *type III edgepath*. For each basic edgepath system $\Lambda = \{\lambda_i\}$, we make an edgepath system Γ that each edgepath γ_i of Γ is a type III edgepath for λ_i. Such an edgepath system is called a *type III edgepath system*.

A candidate edgepath system corresponds to some properly embedded surfaces in the exterior of K as follows. Now, we divide S^3 into N 3-balls so that each ball B_i includes the i-th tangle. Each edgepath γ_i represents a subsurface F_i in the ball B_i whose boundary appears at the tangle in B_i and on the boundary ∂B_i. An edge in an edgepath expresses saddles. A non-constant edgepath corresponds to subsurfaces obtained by combining saddles corresponding to the edges in γ_i. Note here that two possible choices of saddles exist for one edge, and that multiple subsurfaces correspond to an edgepath in general. Besides, a constant edgepath represents a *cap subsurface*. Any candidate edgepath system Γ satisfies conditions that for the endpoints of edgepaths in the edgepath system, u-coordinates coincide and v-coordinates sum up to 0. These conditions assure that subsurfaces $\{F_i\}$ corresponding to edgepaths $\{\gamma_i\}$ are glued consistently into a properly embedded surface F, which is called a *candidate surface*. These conditions are called the condition for *gluing consistency*.

Any essential surface can be isotopically deformed into some standard form, and hence, is isotopic to one of the candidate surfaces (In a precise sense, some essential surfaces are represented by some exceptional edgepath system not described above. However, we can ignore such essential surfaces in this paper.) Conversely, the set of candidate surfaces includes inessential surfaces in general. Some conditions for an edgepath system to correspond to an essential surface are given in [HO89]. With these conditions, by eliminating candidate edgepath systems which give only inessential surfaces, we have the list of edgepath systems corresponding to essential surfaces. By calculating boundary slopes for the essential surfaces, we eventually obtain the list of boundary slopes.

2.3. **Supplements to the main theorem.** For a pretzel knot $K = P(p_1, p_2, \ldots, p_N)$ satisfying (*), a basic edgepath system $\Lambda_A = \{\lambda_{A,i}\}$ whose edgepaths are given by

$$\lambda_{A,i} = \lambda_{i,a} = \langle 0 \rangle - \langle 1/p_i \rangle,$$

is regarded as a type II edgepath system $\Gamma_A = \{\gamma_{A,i}\}$. Though a candidate edgepath system corresponds to multiple candidate surfaces in general, by construction, all the surfaces obtained from Γ_A are isotopic to each other. Let F_A denote the surface so obtained, which is in fact a spanning surface. F_A is obtained by naturally spanning a standard diagram of the pretzel knot K as in the left figure of Figure 3. If the tuple $(p_1, p_2, \ldots, p_N)$ corresponding to K satisfies the condition (a) in Introduction, the surface F_A so constructed is non-orientable. Meanwhile, if the condition (b) is satisfied, the surface F_A becomes orientable.

We define a non-orientable spanning surface F_B as follows: If the condition (a) is satisfied, we set $F_B = F_A$, otherwise F_B is obtained by attaching a half-twisted band to F_A. See Figure 3. By considering the first Betti number of this F_B, we obtain

$$\mathrm{Crosscap}(K) \leq \beta_1(F_B) = \begin{cases} N-1 & \text{when (a) is satisfied,} \\ N & \text{when (b) is satisfied.} \end{cases}$$

Hence, Theorem 1.2 claims that the equality in the inequality above holds; in other words, F_B attains the crosscap number.

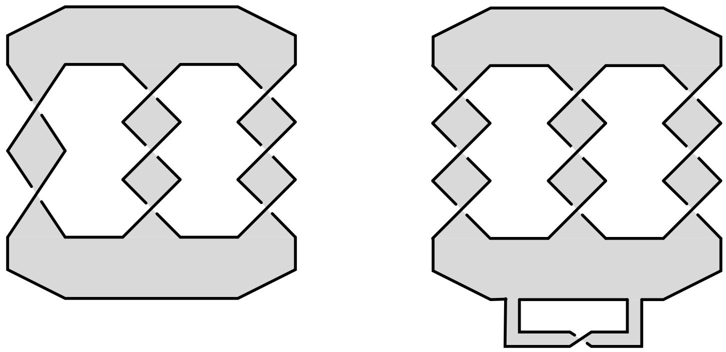

FIGURE 3. $F_A = F_B$ for the $(2,3,3)$-pretzel knot in the class (a) and F_B for the $(3,3,3)$-pretzel knot in the class (b)

Remark 2.1. If $N = 2$, then K is $(2, n)$-cabled and Crosscap$(K) = 1$. Thus Theorem 1.2 holds in this case. If $N = 1$, then K is the trivial knot, and its crosscap number is defined as Crosscap$(K) = 0$. Thus, the expression of Theorem 1.2 is not true when $N = 1$ and the condition (b) with p_1 odd is satisfied.

3. PROOF

As claimed in [IM], in the algorithm of Hatcher and Oertel, the ratio $\frac{-\chi}{\sharp s}(F)$ of the negative $-\chi(F)$ of $\chi(F)$ and $\sharp s(F)$ appears to be more natural for a surface F. Hence we will perform estimation for $\frac{-\chi}{\sharp s}$, and our goal is to give the following.

Proposition 3.1. *Assume that K is a non-trivial pretzel knot $P(p_1, p_2, \ldots, p_N)$ with $N \geq 3$ satisfying (*). Let F_B be the embedded surface in the exterior of K defined in Section 2. Then, for any candidate surface F_E in the exterior of K, the inequality*

$$\frac{-\chi}{\sharp s}(F_B) - 1 \leq \frac{-\chi}{\sharp s}(F_E) \tag{1}$$

holds. Moreover, if the equality holds, F_E does not satisfy the condition "non-orientable and spanning".

Note that $\frac{-\chi}{\sharp s}(F_B)$ is $N-2$ for (a) and $N-1$ for (b). Besides, for a spanning surface F, the first Betti number and the Euler characteristic are related by $\beta_1(F) = -\chi(F) + 1 = \frac{-\chi}{\sharp s}(F) + 1$. With the above proposition, Theorem 1.2 can be proved as follows.

Proof of Theorem 1.2. Since F_B is a non-orientable spanning surface for K, it is clear by definition that Crosscap$(K) \leq \beta_1(F_B)$.

Assume for a contradiction that Crosscap$(K) < \beta_1(F_B)$ holds. Then, for the non-orientable spanning surface F_C which attains the crosscap number, $\beta_1(F_C) =$ Crosscap$(K) \leq \beta_1(F_B) - 1$ holds. Remark that we cannot tell whether F_C is essential or not.

If F_C is essential, then we have $\beta_1(F_B) - 1 \leq \beta_1(F_C)$ by Proposition 3.1. Together with the assumption above, this implies that $\beta_1(F_B) - 1 = \beta_1(F_C)$ must hold. Then, by the condition for the equality in Proposition 3.1, F_C does not satisfy

"non-orientable and spanning". Though, this contradicts to the assumption that F_C is a non-orientable spanning surface.

Now assume that F_C is inessential. As claimed in Lemma 3.1 in [HT], F_C is in fact incompressible. Thus it must be boundary-compressible. By boundary compressions, as argued in the proof of Lemma 3.1 in [HT], an essential surface F_E is obtained. Then it satisfies $-\chi(F_E) < -\chi(F_C)$, and together with $\chi(F_E) \leq 0$ obtained in [IM] and $\sharp s(F_C) = 1 \leq \sharp s(F_E)$, we have $\frac{-\chi}{\sharp s}(F_E) < \frac{-\chi}{\sharp s}(F_C)$. Again, by Proposition 3.1, we have $\frac{-\chi}{\sharp s}(F_B) - 1 \leq \frac{-\chi}{\sharp s}(F_E) < \frac{-\chi}{\sharp s}(F_C)$. It then follows that $\beta_1(F_B) - 1 < \beta_1(F_C)$. Though, it contradicts to $\beta_1(F_C) \leq \beta_1(F_B) - 1$.

Therefore we conclude that $\mathrm{Crosscap}(K) = \beta_1(F_B)$. This completes the proof of the theorem. $\square$

The rest of the paper is devoted to proving Proposition 3.1. In the following, we will assume that the readers are rather familiar with the work of Hatcher and Oertel. Please see [HO89] or [IM] for details.

We here remark that we will not check whether a candidate surface is actually essential or not. That is, we will establish our estimation for all candidate surfaces including both essential and inessential ones.

In the following, we divide the argument into two main cases concerning types of edgepath systems.

3.1. **Type II and type III edgepath systems.** Let us start with considering type II and type III edgepath systems. This case is much easier than the case of type I.

Lemma 3.2. *Assume that K is a non-trivial pretzel knot $P(p_1, p_2, \ldots, p_N)$ with $N \geq 3$ satisfying (*). Let F_B be the embedded surface in the exterior of K defined in Section 2. Then, for any embedded surface F_E in the exterior of K corresponding to a type II or III edgepath system, the inequality*

$$\frac{-\chi}{\sharp s}(F_B) - 1 \leq \frac{-\chi}{\sharp s}(F_E)$$

holds. Moreover, if the equality holds, F_E does not satisfy the condition "non-orientable and spanning".

An edgepath system Γ corresponds to multiple surfaces in general. Though, the value of $\frac{-\chi}{\sharp s}(F)$ is common for any surface F of these surfaces. See [IM] for example. Thus, in the following, we denote the common value by $\frac{-\chi}{\sharp s}(\Gamma)$ for brevity.

Proof of Lemma 3.2. Recall that, for each edgepath γ_i in an edgepath system $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$, the length $|\gamma_{i,>0}|$ of the part $\gamma_{i,>0}$ lying in the region $u > 0$ is one or more, if Γ is of type II or III. On the other hand, a formula of calculating $\frac{-\chi}{\sharp s}(\Gamma)$ for Γ is given as

$$\frac{-\chi}{\sharp s}(\Gamma) \quad = \quad \sum_{i=1}^{N} |\gamma_{i,>0}|$$

when Γ is of type III, and as

$$\frac{-\chi}{\sharp s}(\Gamma) \quad = \quad \left(\sum_{i=1}^{N} (|\gamma_{i,>0}| \right) + V - 2$$

when Γ is of type II. Here V denotes the number of vertical edges.

From these, we immediately have the inequality (1) in Proposition 3.1;

$$\frac{-\chi}{\sharp s}(\Gamma) \geq N-2.$$

The equality can hold only when the condition (b) is satisfied and the edgepath system Γ is of type II with no vertical edges whose all edgepaths have length 1 in the region $u>0$. Note that other than edgepaths of the form $\langle 0\rangle - \langle \pm 1/p\rangle$, the only edgepath $\gamma = \langle \pm 1\rangle - \langle \pm 1/2\rangle$ satisfies $|\gamma| = 1$. However, since we here think about the case where the condition (b) is satisfied, i.e., all p_i's are odd, we do not have to consider $\langle \pm 1\rangle - \langle \pm 1/2\rangle$ for Γ. Therefore the equality holds only when all edgepaths in Γ are of the form $\langle 0\rangle - \langle \pm 1/p\rangle$. This Γ corresponds to the surface F_A, which is orientable when the condition (b) is satisfied. Hence, the condition for the equality in Proposition 3.1 is satisfied. $\square$

3.2. **Type I edgepath systems.** For a type I edgepath system Γ, a formula of calculating $\frac{-\chi}{\sharp s}(\Gamma)$ is given as

$$\frac{-\chi}{\sharp s}(\Gamma) = \sum_{i=1}^{N}\left(\begin{cases} 0 & (\text{ if } \gamma_i \text{ is constant }) \\ |\gamma_i| & (\text{ otherwise }) \end{cases}\right) + N_{\text{const}} - N + \left(N-2-\sum_{\gamma_i\in\Gamma_{\text{const}}}\frac{1}{|p_i|}\right)\frac{1}{1-u}. \tag{2}$$

Here, Γ_{const} denotes the constant edgepaths in Γ, N_{const} is the number of the constant edgepaths, and $|\gamma|$ is the length of an edgepath γ. The length of an edgepath coincides with the path length in the graph if all the edges in the edgepath are complete edges.

Originally, this formula should be applied to the edgepath systems satisfying the gluing consistency. Though, we can apply this to other edgepath systems formally. In this subsection, by using the formula, we will show that $\frac{-\chi}{\sharp s}(\Gamma)$ is bounded from below for all possibilities even if we ignore the gluing consistency in most cases.

3.2.1. *Preparations.* We here introduce certain new functions, a variable and a preorder to clarify and simply the arguments.

First let us introduce a function $Y_\Lambda : (0,1) \longrightarrow \mathbb{R}$ for a fixed basic edgepath system Λ in the following way: We take the extended basic edgepath system $\widetilde{\Lambda}$ for Λ. For $0<u_0<1$, we cut $\widetilde{\Lambda}$ at $u=u_0$ in the manner described in the previous section, and let Γ_{u_0} denote the obtained edgepath system, which is nearly type I but may not satisfy the gluing consistency. Then, we calculate the value $\frac{-\chi}{\sharp s}(\Gamma_{u_0})$ for the edgepath system Γ_{u_0} formally by applying the formula (2). The function $Y_\Lambda(u)$ is defined to take the value $\frac{-\chi}{\sharp s}(\Gamma_{u_0})$ for $u=u_0$.

This function Y_Λ becomes much easier to see by introducing a new variable w and a new function x as follows: We define a new variable w as $w=1/(1-u)$. By this variable transformation, we define a new function X_Λ by $X_\Lambda(w) = Y_\Lambda(1-1/w) = Y_\Lambda(u)$. The function $x_\lambda(w) : [1,\infty) \to \mathbb{R}$ is defined for a fixed basic edgepath λ as follows. We prepare the extended basic edgepath $\widetilde{\lambda}$ for λ. For $w>1$, we obtain an

edgepath $\gamma_{\lambda,w}$ by cutting $\widetilde{\lambda}$ at $u = 1-(1/w)$ in our manner. Then the function x_λ is defined by;

$$x_\lambda(w) \quad = \quad \begin{cases} 1 - \frac{1}{q}w & (\text{ if } \gamma_{\lambda,w} \text{ is a constant edgepath} \\ & \quad \text{with } v\text{-coordinate } p/q) \\ \sum_{j=1}^{\text{number of edges in } \gamma_{\lambda,w}} |e_j| & (\text{ if } \gamma_{\lambda,w} \text{ is non-constant }) \end{cases}$$

where

$$|e_j| \quad = \quad \begin{cases} \frac{s_j - w}{s_j - q_j} \ (= \frac{s_j}{s_j - q_j} - \frac{1}{s_j - q_j}w) & (\text{ if } e_j \text{ is a partial edge of } \langle p_j/q_j \rangle - \langle r_j/s_j \rangle \\ & \quad \text{ending at } u = 1 - 1/w) \\ 1 & (\text{ if } e_j \text{ is a complete edge}) \end{cases} .$$

Note that $x_\lambda(w)$ is a piecewise linear function and is strictly monotonically decreasing. With these w and x_λ, the function X_Λ is expressed simply as follows:

$$X_\Lambda(w) \quad = \quad [(N-2)w - N] + \sum_{i=1}^{N} x_{\lambda_i}(w),$$

where λ_i's are the basic edgepaths in the basic edgepath system Λ. We will always use this expression in actual calculations of $X_\Lambda(w)$ in the sequel. First, we have the following proposition.

Proposition 3.3. *For the $(p_1, p_2, \ldots, p_N)$-pretzel knot K with $N \geq 3$ satisfying (*), let Λ_A be the basic edgepath system $\{\langle 0 \rangle - \langle 1/p_1 \rangle,\ \langle 0 \rangle - \langle 1/p_2 \rangle,\ \ldots,\ \langle 0 \rangle - \langle 1/p_N \rangle\}$. Then, for any basic edgepath system Λ for K, the inequality $X_{\Lambda_A}(w) \leq X_\Lambda(w)$ holds for any $w > 1$.*

Proof. For a rational tangle $1/p_i$, possible basic edgepaths are $\lambda_{i,a} = \langle 0 \rangle - \langle 1/p_i \rangle$ and $\lambda_{i,b} = \langle s_i \rangle - \langle s_i/2 \rangle - \ldots - \langle s_i/(|p_i|-1) \rangle - \langle 1/p_i \rangle)$, where s_i denotes the sign of p_i.

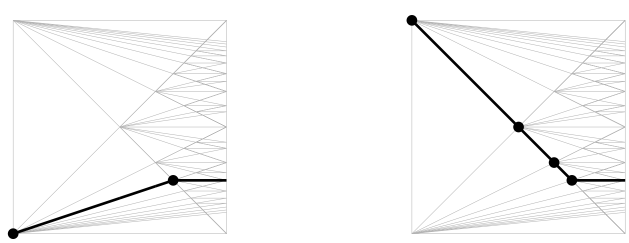

FIGURE 4. $\widetilde{\lambda_{i,a}}$ and $\widetilde{\lambda_{i,b}}$ for $p_i = 4$

The functions x_λ corresponding to these extended basic edgepaths are

$$x_{\lambda_{i,a}}(w) \quad = \quad \begin{cases} 1 - \frac{1}{|p_i|-1}(w-1) & (1 \leq w \leq |p_i|) \\ 1 - \frac{1}{|p_i|}w & (|p_i| < w) \end{cases}$$

and

$$x_{\lambda_{i,b}}(w) \quad = \quad \begin{cases} |p_i| - w & (1 \leq w \leq |p_i|) \\ 1 - \frac{1}{|p_i|}w & (|p_i| < w) \end{cases} .$$

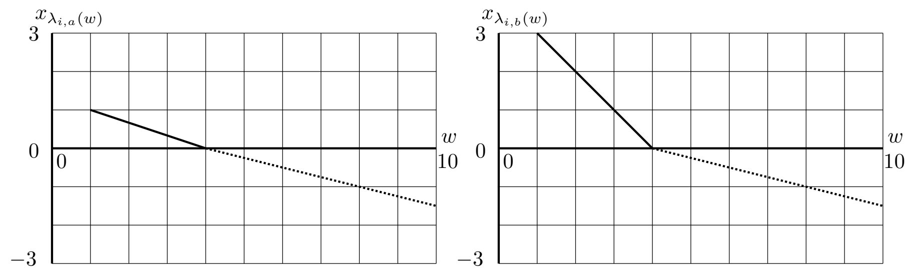

FIGURE 5. $x_{\lambda_{i,a}}(w)$ and $x_{\lambda_{i,b}}(w)$ for $p_i = 4$

With respect to these two edgepaths the inequality $x_{\lambda_{i,a}}(w) \le x_{\lambda_{i,b}}(w)$ clearly holds for any $w > 1$. Hence, for Λ_A, $X_{\Lambda_A}(w)$ takes minimum among X_Λ of all the basic edgepath systems Λ. □

Next we introduce the following preorder for tuples of integers.

Definition 3.4. Let $(p_1, p_2, \ldots, p_n)$ and $(p'_1, p'_2, \ldots, p'_n)$ be n-tuples of integers. Assume that, for these tuples, after replacing each entry by its absolute value, rearrange entries in ascending order, we obtain $(q_1, q_2, \ldots, q_n)$ and $(q'_1, q'_2, \ldots, q'_n)$. Then, if $q_i \le q'_i$ holds for each $i = 1, 2, \ldots, n$, we define $(p_1, p_2, \ldots, p_n) \le (p'_1, p'_2, \ldots, p'_n)$. Since this relation is reflexive and transitive, this gives a preorder. Moreover the binary relation $\sim$ defined as $x \sim y \Leftrightarrow (x \le y \text{ and } y \le x)$ becomes an equivalence relation. Let $|p_1, p_2, \ldots, p_n|$ denote the equivalent class of $(p_1, p_2, \ldots, p_n)$.

For the $(p_1, p_2, \ldots, p_n)$-pretzel knot K and the $(p'_1, p'_2, \ldots, p'_n)$-pretzel knot K', we define a preorder of the pretzel knots by $K \le K' \Leftrightarrow (p_1, p_2, \ldots, p_n) \le (p'_1, p'_2, \ldots, p'_n)$.

With this preorder, we obtain the following proposition naturally.

Proposition 3.5. *Assume that $(p_1,\ p_2,\ \ldots,\ p_N) \le (p'_1,\ p'_2,\ \ldots,\ p'_N)$. Let K and K' be the $(p_1,\ p_2,\ \ldots,\ p_N)$-pretzel knot and the $(p'_1,\ p'_2,\ \ldots,\ p'_N)$-pretzel knot both satisfying (*) respectively. Let Λ_A denote the basic edgepath system $\{\langle 0\rangle - \langle 1/p_1\rangle, \langle 0\rangle - \langle 1/p_2\rangle, \ldots, \langle 0\rangle - \langle 1/p_N\rangle\}$ for the knot K, and Λ' denote an arbitrary basic edgepath system for the knot K'.*

If an inequality $X_{\Lambda_A}(w) > t$ holds for some real number t for any $w > 1$, then the inequality $X_{\Lambda'}(w) > t$ also holds for any $w > 1$. The similar fact holds in the case of $X_{\Lambda_A}(w) \ge t$.

Proof. Let T and T' be the $1/p$ tangle and the $1/p'$ tangle satisfying $|p| \le |p'|$. For two basic edgepaths $\lambda_a = \langle 0\rangle - \langle 1/p\rangle$ and $\lambda'_a = \langle 0\rangle - \langle 1/p'\rangle$ corresponding to T and T' respectively, we immediately have $x_{\lambda_a}(w) \le x_{\lambda'_a}(w)$ for any $w > 1$.

This implies that:

(1) When $(p_1, p_2, \ldots, p_N) \le (p'_1, p'_2, \ldots, p'_N)$, we have: The inequality $X_{\Lambda_A}(w) \le X_{\Lambda'_A}(w)$ holds for any $w > 1$. Here Λ'_A denotes a basic edgepath system $\{\langle 0\rangle - \langle 1/p'_1\rangle, \langle 0\rangle - \langle 1/p'_2\rangle, \ldots, \langle 0\rangle - \langle 1/p'_N\rangle\}$ for the knot K'.
(2) When $|p_1, p_2, \ldots, p_N| = |p'_1, p'_2, \ldots, p'_N|$, we have: The identity $X_{\Lambda_A}(w) = X_{\Lambda'_A}(w)$ holds for any $w > 1$.

Together with Proposition 3.3 and the definition of our preorder, these observations imply the assertion in the proposition. □

Once we have an appropriate lower bound of the function X_{Λ_A} for the edgepath system Λ_A for a pretzel knot K, by the propositions above, we also have the same lower bound of X_Λ for any edgepath system Λ of any pretzel knot K' equal to or greater than K. For any type I edgepath system obtained for these edgepath systems and these pretzel knots, the value of $\frac{-\chi}{\sharp s}$ is bounded by the same bound naturally. Thus, we can establish the required lower bound for all type I edgepath systems of many pretzel knots at the same time, without solving the equation $\widetilde{\Lambda}(u) = 0$. We only have to give a case-by-case argument for only some remaining pretzel knots.

3.2.2. *Type I edgepath systems for $N \geq 4$.* We first consider the number of tangles N is at least 4.

Lemma 3.6. *Assume that K is a non-trivial pretzel knot $K = P(p_1, p_2, \ldots, p_N)$ with $N \geq 4$ satisfying (*). Let F_B be the embedded surface in the exterior of K defined in Section 2. Then, for any embedded surface F_E in the exterior of K corresponding to a type I edgepath system, the inequality*

$$\frac{-\chi}{\sharp s}(F_B) - 1 \leq \frac{-\chi}{\sharp s}(F_E)$$

holds. Moreover, if the equality holds, F_E does not satisfy the condition "non-orientable and spanning".

Proof. Suppose first that $(p_1, p_2, \ldots, p_N)$ satisfies the condition (a). For the pretzel knot K with $(p_1, p_2, \ldots, p_N) = (2, 3, \ldots, 3)$, and the basic edgepath system $\Lambda_A = \{\langle 0\rangle - \langle 1/2\rangle,\ \langle 0\rangle - \langle 1/3\rangle,\ \ldots,\ \langle 0\rangle - \langle 1/3\rangle\}$, the function $X_{\Lambda_A}(w)$ is a piecewise linear function described as; $N-2 (> N-3)$ for $w = 1$, $(N-5)w/2 + (N+1)/2$ for $1 \leq w \leq 2$, $(3N-9)/2$ $(= N-3+(N-3)/2 > N-3)$ for $w = 2$, $(N-4)w/2+(N-1)/2$ for $2 \leq w \leq 3$, $(4N-13)/2$ $(= N-3+(2N-7)/2 > N-3)$ for $w = 3$, $(4N-13)w/6$ for $3 \leq w$, and goes to ∞ when w goes to ∞. Clearly, $X_{\Lambda_A}(w) > N-3$ holds for any $w > 1$.

For any $(p_1, p_2, \ldots, p_N)$-pretzel knot in this case, that is, the conditions (*) and (a) are satisfied and $N \geq 4$, $(p_1, p_2, \ldots, p_N)$ is equal to or greater than $(2, 3, \ldots, 3)$. Therefore F_E always satisfies

$$\frac{-\chi}{\sharp s}(F_B) - 1 = N - 3 < \frac{-\chi}{\sharp s}(F_E)$$

by Proposition 3.5.

Suppose next that $(p_1, p_2, \ldots, p_N)$ satisfies the condition (b). Since N must be odd, we remark that $N \geq 5$. For the pretzel knot K with $(p_1, p_2, \ldots, p_N) = (3, 3, \ldots, 3)$, and the basic edgepath system $\Lambda_A = \{\langle 0\rangle - \langle 1/3\rangle,\ \langle 0\rangle - \langle 1/3\rangle,\ \ldots, \langle 0\rangle - \langle 1/3\rangle\}$, the function $X_{\Lambda_A}(w)$ is a piecewise linear function described as; $N-2$ for $w = 1$, $(N-4)w/2 + N/2$ for $1 \leq w \leq 3$, $2N-6$ $(= N-2+(N-4) > N-2)$ for $w = 3$, $(2N-6)w/3$ for $3 \leq w$ and goes to ∞ when w goes to ∞. Clearly, $X_{\Lambda_A}(w) > N-2$ holds for any $w > 1$. For any $(p_1, p_2, \ldots, p_N)$-pretzel knot in this case, that is, the conditions (*) and (b) are satisfied and $N \geq 5$, $(p_1, p_2, \ldots, p_N)$ is equal to or greater than $(3, 3, \ldots, 3)$. Therefore F_E always satisfies

$$\frac{-\chi}{\sharp s}(F_B) - 1 = N - 2 < \frac{-\chi}{\sharp s}(F_E)$$

by Proposition 3.5. □

3.2.3. *Type I edgepath systems for $N = 3$.* The case $N = 3$ only remains, and we treat it in the next lemma.

Lemma 3.7. *Assume that K is a non-trivial pretzel knot $K = P(p_1, p_2, p_3)$ satisfying (*). Let F_B be the embedded surface in the exterior of K defined in Section 2. Then, for any embedded surface F_E in the exterior of K corresponding to a type I edgepath system, the inequality*

$$\frac{-\chi}{\sharp s}(F_B) - 1 \leq \frac{-\chi}{\sharp s}(F_E)$$

holds. Moreover, if the equality holds, F_E does not satisfy the condition "non-orientable and spanning".

Proof. Suppose first that (p_1, p_2, p_3) satisfies the condition (a). If K is $(-2, 3, 3)$ or $(-2, 3, 5)$-pretzel knot, then it is a torus knot. In this case, as in [IM], an annulus with $\frac{-\chi}{\sharp s} = 0 = N - 3$ is obtained. However, since the number of sheets $\sharp s = 2$ is greater than 1, the condition for the equality is satisfied. Hence, this case does not matter. For the other candidate surfaces,

$$N - 3 = 0 < 1 \leq \frac{-\chi}{\sharp s}(F_E)$$

holds as in [IM]. In fact, Crosscap$(K) = 2$ can be obtained also as the result in ([T04]) about torus knots. If K is one of the other pretzel knots in this case, by the result given in [IM], all candidate surfaces satisfy

$$N - 3 = 0 < \frac{-\chi}{\sharp s}(F_E).$$

Suppose next that (p_1, p_2, p_3) satisfies the condition (b). Similarly in the proof of Lemma 3.6, we prepare Λ_A for a (p_1, p_2, p_3)-pretzel knot satisfying $|p_1, p_2, p_3| = |3, 5, 7|$ or $|p_1, p_2, p_3| = |5, 5, 5|$. By concrete calculation of the function X_{Λ_A}, we have the following: For the pretzel knots with $|3, 5, 7|$, the function $X_{\Lambda_A}(w)$ is a piecewise linear function expressed as; 1 for $w = 1$, $w/12 + 11/12$ for $1 \leq w \leq 3$, $7/6 (> 1)$ for $w = 3$, $w/4 + 5/12$ for $3 \leq w \leq 5$, $5/3 (> 1)$ for $w = 5$, $3w/10 + 1/6$ for $5 \leq w \leq 7$, $34/15 (> 1)$ for $w = 7$, $34w/105$ for $7 \leq w$, and goes to ∞ when w goes to ∞. Similarly, for the pretzel knot with $|5, 5, 5|$, the function $X_{\Lambda_A}(w)$ is expressed as; 1 for $w = 1$, $w/4 + 3/4$ for $1 \leq w \leq 5$, $2 (> 1)$ for $w = 5$, $2w/5$ for $5 \leq w$, and goes to ∞ when w goes to ∞. The graphs of the functions $X_{\Lambda_A}(w)$ are illustrated in Figure 6. Clearly, $X_{\Lambda_A}(w) > 1$ holds for any $w > 1$. Therefore, if K is equal to or greater than either of these two knots with respect to our preorder, and if F_E corresponds to a type I edgepath system, we have

$$\frac{-\chi}{\sharp s}(F_B) - 1 = N - 2 = 1 < \frac{-\chi}{\sharp s}(F_E)$$

by Proposition 3.5.

The case of $|p_1, p_2, p_3| = |3, 3, n|$ with $n \geq 3$ and $|p_1, p_2, p_3| = |3, 5, 5|$ are only remaining. Even for these cases, by [IM],

$$\frac{-\chi}{\sharp s}(F_B) - 1 = N - 2 = 1 \leq \frac{-\chi}{\sharp s}(F_E)$$

holds. Hence, only the surfaces satisfying the equality in the proposition matter. We investigate these cases independently as follows.

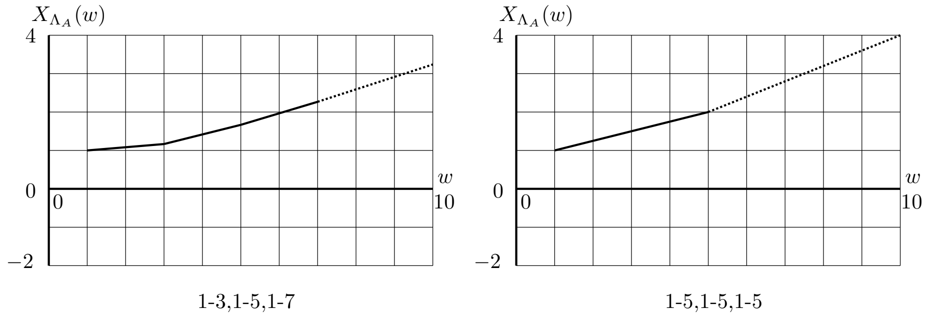

FIGURE 6. The graphs of the functions $X_{\Lambda_A}(w)$ for Λ_A of pretzel knots with $|3,5,7|$ and $|5,5,5|$

Note here that for (p_1, p_2, p_3) and $(p'_1, p'_2, p'_3) = (-p_1, -p_2, -p_3)$, there exists a one-to-one correspondence between candidate surfaces for the former and candidate surfaces for the latter preserving the number $\sharp s$ of sheets and the Euler characteristic χ. Similarly, if (p'_1, p'_2, p'_3) is obtained from (p_1, p_2, p_3) by exchanging p_i and p_j, a similar correspondence exists. Besides, if all p_i's have the common sign, by the gluing consistency, there exist no type I edgepath systems. Eventually, it suffices to show for the cases of $(p_1, p_2, p_3) = (-3, \pm 3, n)$ and $(-3, \pm 5, 5)$. One more important fact here is that if some edgepath in an edgepath system includes a partial edge of a non-horizontal edge, then the number $\sharp s$ of sheets must be greater than 1, that is, the corresponding surface is not spanning.

- $(p_1, p_2, p_3) = (-3, -3, n)$:
  The edgepath system $\Gamma = \{\ ((1/2)\langle 0\rangle + (1/2)\langle -1/3\rangle) - \langle -1/3\rangle,\ ((1/2)\langle 0\rangle + (1/2)\langle -1/3\rangle) - \langle -1/3\rangle,\ \langle 1/2\rangle - \langle 1/3\rangle - \ldots - \langle 1/(n-1)\rangle - \langle 1/n\rangle\}$ is the unique type I edgepath system, and we have $\frac{-\chi}{\sharp s}(\Gamma) = n - 2 \geq 1$. However, since $\sharp s > 1$, the corresponding surface is not a spanning surface.
- $(p_1, p_2, p_3) = (-3, 3, n)$:
  The edgepath system $\Gamma = \{\ ((4/(n+1))\langle -1/2\rangle + ((n-3)/(n+1))\langle -1/3\rangle) - \langle -1/3\rangle,\ ((2/(n+1))\langle 0\rangle + ((n-1)/(n+1))\langle 1/3\rangle) - \langle 1/3\rangle,\ (((n-1)/(n+1))\langle 0\rangle + (2/(n+1))\langle 1/n\rangle) - \langle 1/n\rangle\}$ is the unique type I edgepath system, and we have $\frac{-\chi}{\sharp s}(\Gamma) = 1 = N - 2$. However, since $\sharp s > 1$, the corresponding surface is not a spanning surface.
- $(p_1, p_2, p_3) = (-3, -5, 5)$:
  The edgepath system $\Gamma = \{\ ((1/3)\langle 0\rangle + (2/3)\langle -1/3\rangle) - \langle -1/3\rangle,\ ((2/3)\langle 0\rangle + (1/3)\langle -1/5\rangle) - \langle -1/5\rangle,\ ((2/3)\langle 1/2\rangle + (1/3)\langle 1/3\rangle) - \langle 1/3\rangle - \langle 1/4\rangle - \langle 1/5\rangle\}$ is the unique type I candidate edgepath system, and we have $\frac{-\chi}{\sharp s}(\Gamma) = 3 > 1$.
- $(p_1, p_2, p_3) = (-3, 5, 5)$:
  Only the family of edgepath systems $\Gamma = \{\ (((2-3u)/(2-2u))\langle 0\rangle + (u/(2-2u))\langle -1/3\rangle) - \langle -1/3\rangle,\ (((4-5u)/(4-4u))\langle 0\rangle + (u/(4-4u))\langle 1/5\rangle) - \langle 1/5\rangle,\ (((4-5u)/(4-4u))\langle 0\rangle + (u/(4-4u))\langle 1/5\rangle) - \langle 1/5\rangle\}$ for $0 < u < 2/3$ are obtained from this pretzel knot as type I edgepath systems. These edgepath systems correspond to the non-isolated solution of the gluing consistency,

and we have $\frac{-\chi}{\sharp s}(\Gamma) = 1$ for any of the edgepath systems. $0 < 1/2 \leq (4-5u)/(4-4u) < 1$ means $\sharp s > 1$, and so, any corresponding surface is not a spanning surface.

This completes the proof of the lemma. □

Consequently, Proposition 3.1 follows from Lemmas 3.2, 3.6 and 3.7.

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